MR1743970 (2001j:17011) 17Bxx 17-01 17-02 17-08 68W30
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★**Lie algebras: theory and algorithms.**
North-Holland Mathematical Library, 56.
North-Holland Publishing Co., Amsterdam, 2000. *xii*+393 *pp.* \$118.00.
ISBN 0-444-50116-9

This book presents both the Lie algebra theory and algorithms for computing much of the basic structure of Lie algebras. It is one of the first books which combines both these aspects in one volume. The book is well written and the material is clearly presented.

Besides the usual Lie algebra theory for algebraically closed fields of characteristic zero the book also considers Lie algebras over non-algebraically closed fields and the restricted Lie algebras in characteristic p. The results for all of these are developed simultaneously and the results for the special cases are interwoven throughout the text. Each chapter starts by developing the Lie algebra theory, and algorithms are presented following the theoretical results. Most of the algorithms can be easily implemented in several of the existing computer algebra packages currently available, such as LiE, GAP4 and Magma, and in fact many of these algorithms have already been included in these programs.

The book consists of eight chapters and an appendix on associative algebras. At the end of each chapter there is a notes section, which gives references to the pertinent literature.

Chapter 1 presents some of the basic concepts and constructions used in the remainder of the book. This includes associative Lie algebras, examples of the classical Lie algebras, structure constants, quotient algebras, morphisms and automorphisms of Lie algebras, centralizers and normalizers, chains of ideals, derivations, representations, restricted Lie algebras, direct sum decompositions, extensions of the ground field and, of course, algorithms for many of these. Chapter 2 develops the basic theory of nilpotent and solvable Lie algebras. This includes Lie's and Engel's theorems, Cartan's criterion for solvability, a characterization of the solvable radical and nilradical, and a method for finding non-nilpotent elements.

Chapter 3 discusses Cartan subalgebras and related decompositions, such as the primary decomposition and the Fitting decomposition. I was glad to see that Cartan subalgebras were introduced early on in the book, as this emphasizes their central role in the study of Lie algebras and their representations. In most Lie algebra texts they are usually introduced much later, at the very least after a discussion of semisimple or reductive Lie algebras. The book introduces Cartan subalgebras for arbitrary Lie algebras and also considers the case of the so-called split Cartan subalgebras H in a Lie algebra L, which are those Cartan subalgebras for which the eigenvalues of $\mathrm{ad}_L h$, with $h \in H$, are contained in the ground field F. In this case the primary decomposition is also called the root space decomposition. This chapter also discusses regular elements and the conjugacy of Cartan subalgebras in the case of Lie algebras defined over an algebraically closed field of characteristic zero. Other results in the case when the ground field is of characteristic zero include the conjugacy of Cartan subalgebras of solvable Lie algebras and a computation of the nilradical.

Chapter 4 focusses on Lie algebras with nondegenerate Killing forms. This nondegeneracy of the Killing form implies that the Lie algebra is semisimple, so, in particular, the center is zero. In the case when the ground field is algebraically closed of characteristic zero the converse of this result also holds. This result is also known as Cartan's criterion for semisimplicity. The book also discusses the direct sum decomposition into simple ideals, complete reducibility of representations (Weyl's theorem), derivations, Jordan decomposition for Lie algebras, Levi's theorem in the case when the characteristic of the ground field is zero, existence of Cartan subalgebras and a thorough discussion of the properties of the roots and the root space decomposition in the case of split Cartan subalgebras. This chapter also gives some proofs of these properties of the roots for modular fields of characteristic not equal to 2 or 3 and a thorough discussion of the relation of Cartan subalgebras and Levi subalgebras.

Chapter 5 is devoted to the classification of the simple Lie algebras defined over a field of characteristic zero. This chapter starts with a discussion of the representations of $\mathfrak{sl}_2(F)$ and derives thereby some additional properties of the roots; in particular, it is shown that they form a root system. The book proceeds to classify these root systems, starting with the rank-2 root systems, then moving on by introducing bases, Cartan matrices, Weyl groups, Dynkin diagrams, and finally a classification of the Dynkin diagrams and the construction of the corresponding root systems. The isomorphism theorem is proved in the section after that. The remainder of this chapter is devoted to constructing the simple Lie algebras from the root systems, starting with the simply laced ones.

Chapter 6 discusses a number of results about the universal enveloping algebra. First, Gröbner bases for ideals in associative algebras are introduced and are then used to prove the Poincaré-Birkhoff-Witt theorem. From this it follows that there is a one-to-one correspondence between the representations of the Lie algebra and the representations of its universal enveloping algebra. Next a criterion is given to compute a Gröbner basis. The remainder of this chapter discusses an algorithm for constructing faithful finite-dimensional representations of Lie algebras in the case that the characteristic is zero. This also leads to proofs of Ado's and Iwasawa's theorems, which show the existence of such representations.

Chapter 7 contains a discussion of free Lie algebras and their subclass of finitely presented Lie algebras. After introducing both of these algebras, the book gives an algorithm for finding various bases for these Lie algebras. For this Hall sets are used. This chapter concludes with a proof of a theorem by J.-P. Serre which describes the semisimple Lie algebras of characteristic 0 as finitely presented Lie algebras.

The last chapter is devoted to a study of finite-dimensional representations of Lie algebras with a split Cartan subalgebra in the case when the characteristic is zero. Among the topics discussed in this section are highest-weight modules, Verma modules, integral functions, orbits of the Weyl group, Freudenthal's multiplicity formula, Weyl's character, and dimension formulas, and also formulas due to Kostant, Racah and Steinberg.

Finally, the book also contains an appendix which discusses some generalities about associative algebras.

Overall this book is a welcome addition to the literature on Lie algebras and their representations. It also gives an excellent exposition of many of the current Lie algebra algorithms. It is well written and well suited as a textbook for an advanced graduate course in Lie algebras and their computation. It is, of course, also a good reference book for those who need to find algorithms for computations in Lie algebras.

One flaw of the book is its index of terminology. Many of the pages referred to in the index of terminology are incorrect; sometimes the numbers are off by as much as 50 pages. This makes the book harder to use as a reference work. This is something the editor should have noticed before the book went into print. I hope this will be corrected in the second edition. *Aloysius Helminck*